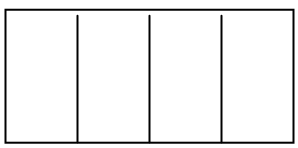
AP Calculus. Section 4.4 OPTIMIZATION

Your chances for success are ___________________________!

Set up an equation in terms of one variable for each of the following problems, then find the maximum.

1. A rectangular wire fence to keep horses contained is to border a river. No fencing is necessary along the river. If 600 m of fencing is available, write an equation for the area of the rectangle.

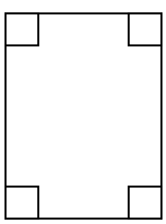
2. A small motel is to be built as shown in the sketch with two long walls y feet each and 6 short walls x feet long each. The total length of the walls is to be 300 feet. Write an equation for the square feet of area taken up by the motel.



3. Two numbers add to 30, write an equation for the product of the two numbers.

4. A rectangle has its base on the x-axis and its upper vertices on the parabola on the parabola $y = -x^2 + 20$. Write an equation for the area of the rectangle.

5. Given a sheet of paper, 8.5 inches by 11 inches. Create a box by cutting from each corner a square of dimension x by x and folding up along the dotted lines. Write an equation for the volume of the box.



6. You want to buy Mrs. DiMarco the largest possible birthday gift, but only have 600 in^2 of wrapping paper. Assuming her gift will fit in a rectangular box with a square bottom, what is the largest volume present you can buy her? (Do not use proper wrapping methods – just cover the outside with paper!)

7. You found the perfect gift for Mrs. DiMarco! It is ______________________________________, which you decide will look best in a right circular cylindrical container. If the volume of the container must be 700 in^3, what is the least amount of wrapping paper you can use to wrap your amazing gift? (Assume no paper waste.)

8. You are to design a cylindrical can that costs the least, with the following constraints:
a) The can must contain 355 cm^3
b) The wall material costs .2 cents per cm^2
c) The top and bottom cost .4 cents per cm^2
What are the dimensions of such a can?

9. A piece of wire that is 10 meters long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area is a:
a) maximum?
b) minimum?

10. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 4.

11. A rectangle storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice its width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.

12. You are designing a rectangular poster to contain 50 square inches of printing with a 4-in margin at the top and bottom and a 2 in margin on each side. What overall dimensions will minimize the amount of paper used?

13. You have been asked to design a 1000 cubic centimeter cylindrical can. What dimensions of the can will require the least amount of material to make it? (Hint: What are we going to minimize?)

14. Mrs. DiMarco is in a boat 3 miles offshore and wants to meet up with Mr. Frees and Ms. Orloff in a village 7 miles down a straight shoreline from the point nearest the boat. Mrs. DiMarco is really strong, but slow walking, so she can row 3 mph and can walk 4 mph. Where should she land the boat to reach her two colleagues in the least amount of time?

15. Suppose that DiMarco, Inc. has a revenue of $r(x) = 9x$ and has a cost of $c(x) = x^3 - 6x^2 + 15x$ where x represents thousands of units of math facts (yes, there are that many). Is there a production level that maximizes profit? If so, what is it? Is there a production level that minimizes average cost? If so, what is it?

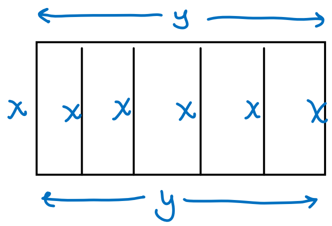
16. A number plus twice a second number is 108. Find the two numbers that give a maximum product. Justify your answer.

17. A rectangular solid that has a square base has a surface area of 150 square inches. Find the maximum volume of the solid and its dimensions. Justify your answer.

18. The sum of the perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a minimum total area. Justify your answer.

19.The diameter plus the height of a cylindrical package is equal to 108 inches. Find the dimensions of the package that gives you a maximum volume. Justify your answer

2. A small motel is to be built as shown in the sketch with two long walls y feet each and 6 short walls x feet long each. The total length of the walls is to be 300 feet. Write an equation for the square feet of area taken up by the motel. Find the maximum area of the motel.



2. Setup: $\boxed{6x + 2y = 300}$ $x: (0, 50)$

$3x + y = 150$

$y = 150 - 3x$

$A = xy$

$A = x(150 - 3x)$

$\boxed{A = 150x - 3x^2}$

2. Solution: $A' = 150 - 6x$

Crit Pts: $150 - 6x = 0$

$x = 25$

$A'' = -6$

$x = 25$ is location of max bc $A'(25) = 0$ and $A''(25) < 0$.

Max area $= A(25) = 25(150 - 3(25)) = \boxed{1875 \text{ ft}^2}$

3. Two numbers add to 30, write an equation for the product of the two numbers. Find the maximum the product can be.

3. Setup: $\boxed{x+y=30}$ $\qquad x: [0, 30]$

$$y = 30 - x$$

$$P = xy$$

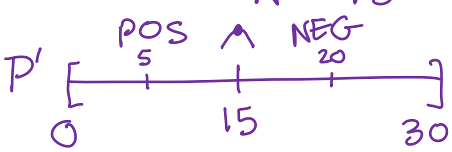
$$P = x(30-x)$$

$$\boxed{P = 30x - x^2}$$

3. Solution: $P' = 30 - 2x$

Crit Pts: $30 - 2x = 0$

$$x = 15$$

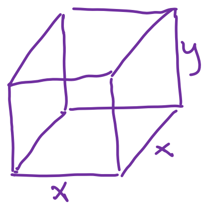


$x=15$ is location of max bc P' changes from pos to neg @ $x=15$ (and it is only max in interval.)

$P(15) = 15(30-15) = \boxed{225}$

6. You want to buy Mrs. DiMarco the largest possible birthday gift, but only have 600 in^2 of wrapping paper. Assuming her gift will fit in a rectangular box with a square bottom, what is the largest volume present you can buy her? (Do not use proper wrapping methods – just cover the outside with paper!)

6. Setup:



Surface Area $= 600$ in^2

$$\boxed{2x^2 + 4xy = 600}$$

$x : (0, \sqrt{300})$

$$4xy = 600 - 2x^2$$

$$y = \frac{600 - 2x^2}{4x}$$

$$y = \frac{300 - x^2}{2x}$$

$$\text{Volume} = x^2 y$$

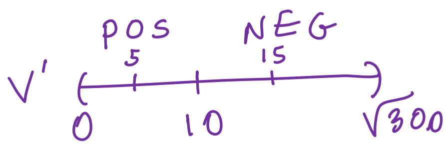
$$V = x^2\left(\frac{300 - x^2}{2x}\right)$$

$$V = \frac{300x - x^3}{2} = 150x - \frac{1}{2}x^3$$

6. Solution: $V' = 150 - \frac{3}{2}x^2$

Crit Pts: $150 - \frac{3}{2}x^2 = 0$

$$x = \pm 10$$



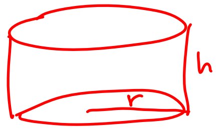
$x = 10$ is location of max bc V' changes from pos to neg at $x = 10$

(and $x = 10$ is location of only max in interval).

Max Vol $= V(10) = 150(10) - \frac{1}{2}(10)^3 = \boxed{1000 \text{ in}^3}$

7. You found the perfect gift for Mrs. DiMarco! It is ____money!____________________________, which you decide will look best in a right circular cylindrical container. If the volume of the container must be 700 in^3, what is the least amount of wrapping paper you can use to wrap your amazing gift? (Assume no paper waste.)

7. Setup:

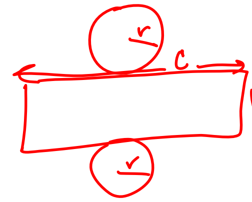


$V = \pi r^2 h$

$\boxed{700 = \pi r^2 h}$

$h = \frac{700}{\pi r^2}$

$r: \left(0, \sqrt{\frac{700}{\pi}}\right)$



$S.A._{cyl} = 2\pi r^2 + 2\pi r h$

$A = 2\pi r^2 + 2\pi r\left(\frac{700}{\pi r^2}\right)$

$\boxed{A = 2\pi r^2 + \frac{1400}{r}}$

7. Solution: $\frac{dA}{dr} = 4\pi r - \frac{1400}{r^2}$

Crit pts: und @ $r = 0$

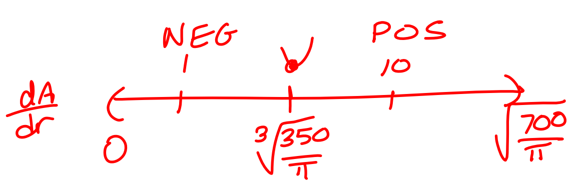
$0 = 4\pi r - \frac{1400}{r^2}$

$\frac{1400}{r^2} = 4\pi r$

$1400 = 4\pi r^3$

$r^3 = \frac{350}{\pi}$

$r = \sqrt[3]{\frac{350}{\pi}}$



$r = \sqrt[3]{\frac{350}{\pi}}$ is location of min bc $\frac{dA}{dr}$ changes from neg to pos @ $r = \sqrt[3]{\frac{350}{\pi}}$ (and it is location of only min in interval)

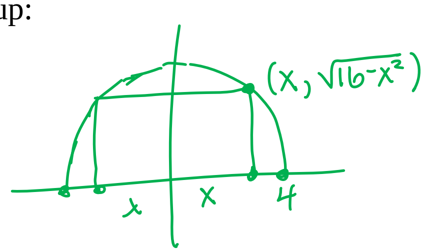
Min S.A. $= \boxed{\left[2\pi\left(\sqrt[3]{\frac{350}{\pi}}\right)^2 + 2\pi\left(\sqrt[3]{\frac{350}{\pi}}\right)\left(\frac{700}{\pi\left(\sqrt[3]{\frac{350}{\pi}}\right)^2}\right)\right] \text{in}^2}$

10. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 4.

10. Setup:

$x^2 + y^2 = 16$ equation of circle

$y = \sqrt{16-x^2}$



$x: (0, 4)$

$A = 2xy$

$$\boxed{A = 2x\sqrt{16-x^2}}$$

10. Solution: $\frac{dA}{dx} = 2\sqrt{16-x^2} + 2x\left(\frac{1}{2}(16-x^2)^{-\frac{1}{2}}(-2x)\right)$ (product rule, chain rule)

$$\frac{dA}{dx} = 2\sqrt{16-x^2} - \frac{2x^2}{\sqrt{16-x^2}}$$

Undefined for $(-\infty,-4] \cup [4,\infty)$

$$0 = 2\sqrt{16-x^2} - \frac{2x^2}{\sqrt{16-x^2}}$$

$$\frac{2x^2}{\sqrt{16-x^2}} = 2\sqrt{16-x^2}$$

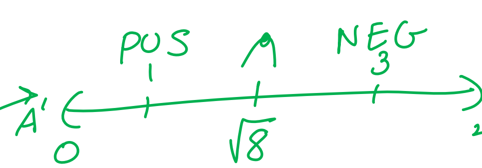
$2x^2 = 2(16-x^2)$

$2x^2 = 32 - 2x^2$

$4x^2 = 32$

$x^2 = 8$

$x = \pm\sqrt{8}$

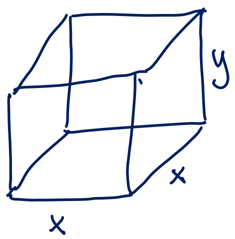


$x=\sqrt{8}$ is location of max bc A' goes from pos to neg @ $x=\sqrt{8}$ (and it is only max in interval).

Max $A = 2\sqrt{8}\left(\sqrt{16-(\sqrt{8})^2}\right) = 2\sqrt{8}(\sqrt{8}) = \boxed{16\ u^2}$

17. A rectangular solid that has a square base has a surface area of 150 square inches. Find the maximum volume of the solid and its dimensions. Justify your answer.

17. Setup:



$S.A. = 2x^2 + 4xy = 150$

$x: (0, \sqrt{75})$

$4xy = 150 - 2x^2$

$y = \dfrac{150-2x^2}{4x}$

$y = \dfrac{75-x^2}{2x}$

$V = x^2 y$

$V = x^2\left(\dfrac{75-x^2}{2x}\right)$

$\boxed{V = \dfrac{75}{2}x - \dfrac{1}{2}x^3}$

17. Solution: $\dfrac{dV}{dx} = \dfrac{75}{2} - \dfrac{3}{2}x^2$

Crit Pts: Not undefined,

$0 = \dfrac{75}{2} - \dfrac{3}{2}x^2$

$\dfrac{3}{2}x^2 = \dfrac{75}{2}$

$x^2 = 25$

$x = \pm 5$

$\dfrac{d^2V}{dx^2} = -3x$

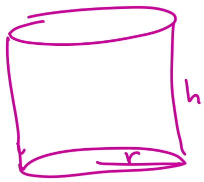
$\left.\dfrac{d^2V}{dx^2}\right|_{x=5} = -3(5) = -15$

$x = 5$ is location of max bc $\left.\dfrac{dV}{dx}\right|_{x=5} = 0$ and $\left.\dfrac{d^2V}{dx^2}\right|_{x=5} < 0$, and it is only max in interval.

$\text{Max } V = (5)^2\left(\dfrac{75-5^2}{2(5)}\right) = \boxed{125 \text{ in}^3}$

19.The diameter plus the height of a cylindrical package is equal to 108 inches. Find the dimensions of the package that gives you a maximum volume. Justify your answer

19. Setup:



$d + h = 108$

$2r + h = 108$

$h = 108 - 2r$

$r: (0, 54)$

$V = \pi r^2 h$

$V = \pi r^2(108 - 2r)$

$V = 108\pi r^2 - 2\pi r^3$

19. Solution: $\frac{dV}{dr} = 216\pi r - 6\pi r^2$

crit pts: not undefined, $\frac{dV}{dr} = 0$

$0 = 216\pi r - 6\pi r^2$

$0 = 6\pi r(36 - r)$

$r = \cancel{0}, 36$

$\frac{d^2V}{dr^2} = 216\pi - 12\pi r$

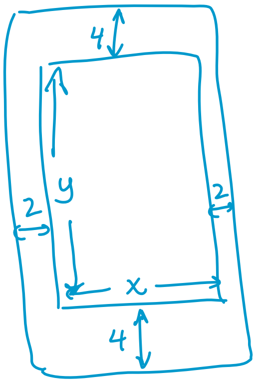
$\left.\frac{d^2V}{dr^2}\right|_{r=36} = 216\pi - 12\pi(36) = -216\pi$

$r = 36$ is radius that gives max volume b/c $\left.\frac{dV}{dr}\right|_{r=36} = 0$ and $\left.\frac{d^2V}{dr^2}\right|_{r=36} < 0$, and it is only max in interval $(0, 54)$.

$\text{Max } V = 108\pi(36)^2 - 2\pi(36)^3 = \boxed{46{,}656\pi \text{ in}^3}$

12. You are designing a rectangular poster to contain 50 square inches of printing with a 4-in margin at the top and bottom and a 2 in margin on each side. What overall dimensions will minimize the amount of paper used?

12. Setup:



$xy = 50$

$y = \frac{50}{x}$

Total $A. = (x+4)(y+8)$

$A = (x+4)\left(\frac{50}{x} + 8\right)$

$A = 50 + 8x + \frac{200}{x} + 32$

$A = 82 + 8x + \frac{200}{x}$

$x: (0, 50)$

12. Solution: $\frac{dA}{dx} = 8 - \frac{200}{x^2}$

Crit Pts: Und. at $x=0$

$0 = 8 - \frac{200}{x^2}$

$\frac{200}{x^2} = 8$

$8x^2 = 200$

$x^2 = 25$

$x = \pm 5$

$\frac{dA}{dx}$: NEG, POS; 0, 1, 5, 10, 50

$x=5$ is minimum dimension bc $\frac{dA}{dx}$ changes signs from neg to pos @ $x=5$, and it is only minimum in interval.

Min dimensions: 9×18